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CREEP OF AN ICE COATING LYING UPON A HYDRAULIC FOUNDATION UNDER THE ACTION OF A CONCENTRATED FORCE

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UDC 624.131+539.215

We consider an infinitely thin plate under cylindrical quasistatic inflection conditions lying upon a hydraulic foundation, i.e., a layer of viscous liquid of finite depth. The plate material is incompressible and such that the intensity of the deformation rate deviator and the stress deviator are related by a power dependence. Such relationships are often used to describe the stress-deformed state of constructions of ice under conditions of developed steady state creep. In the case of plate inflection by a concentrated force asymptotic expressions are found for deflection at short and long times. The solutions constructed make it possible to find the dependence of the deflection of an ice coating upon time as found in full-scale studies.

1. Formulation of the Problem. We will consider the case of cylindrical inflection of a plate of thickness h, lying upon a layer of incompressible viscous liquid of finite depth H (Fig. 1). The inflection is accomplished by a concentrated force Q. To perform experiments on ice covers a special ice-cutting apparatus was used, weighing 2.5 tons, about 7 m long, with a bearing surface of 0.13 m^2, and additional load weights from 2 to 30 tons. The thickness of the ice plate h = 0.25-0.35 m, with the thickness of the liquid layer H = 10-16 m.

We assume that the plate material can be described by Glen's law, which assumes a power relationship between the intensity of tangent stresses and the deformation rate [1], $\dot{\varepsilon} = B\sigma^b$, where B and b are a coefficient and the creep index (greater than unity). In solving the problem we will use the steady state creep equation, i.e., the plate's deflection should increase linearly with time. In reality, because of the interaction between the plate and the base its deflection proves to depend nonlinearly on time, but the linear term is the main one at both short and long relative time intervals.

The deflection of the plate can always be expressed in the form $y = y_e + y_c$, where the term y_e considers elastic and plastic deformations which develop practically instantaneously upon load application, while y_c de-

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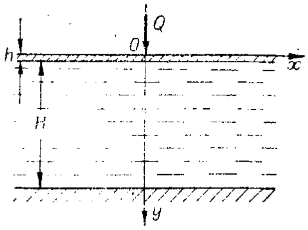

Fig. 1

scribes the deformation which results because of creep of the material. Consideration of the first term y_e as compared to the developed creep deformation y_c is unnecessary, only complicating the problem without affecting the result. Neglect of elastic and plastic deformation components with respect to well developed creep is a generally accepted approach.

We will model the hydraulic base by a viscoelastic Winkler base. Then the reactive pressure acting on the plate from the direction of the liquid will be described by [2]

$$P = ly + m\dot{y}. \tag{1.1}$$

The problem of inflection of the plate with gradual hardening leads to a fourth order differential equation [3]

$$P = K[(v'')^{\mu}]'', \quad v = \dot{y}. \tag{1.2}$$

Comparing Eqs. (1.1) and (1.2) we find that for areas of the plate not acted upon by the concentrated force a solution must be found for the differential equation

$$K[(\dot{y}'')^{\mu}]'' + m\dot{y} + ly = 0 \tag{1.3}$$

for zero initial condition $y|_{t=0} = 0$ with the further condition that the deflection and its first derivative vanish at infinity

$$y(\pm\infty) = 0; \tag{1.4}$$

$$y'(\pm\infty) = 0. \tag{1.5}$$

In Eq. (1.3) the coefficients K and μ characterize the rheological properties of the plate material and are related to the rheological constants in Glen's law: $\mu = 1/b$, $K = gD_{\mu}$, where $D_{\mu} = B^{-\mu}h^{\mu+2}/2^{\mu}(2+\mu)$ corresponds to the rigidity of the plate with respect to inflection under steady state creep conditions [1].

The coefficient m characterizes the resistance to flow of the liquid beneath the plate due to increase in time of the plate deflection, i.e., it characterizes the viscous properties of the Winkler base. Despite the slow increase in creep deformation of the plate, we feel that consideration of the base viscosity is significant, since it permits satisfaction of the conditions at infinity, Eqs. (1.4), (1.5). It is assumed that for a relatively small depth H (since the experiments were performed in shoal waters) the coefficient m is inversely proportional to the magnitude of H and directly proportional to the liquid density ρ and its viscosity ν. Thus $m = \nu\rho/H$. For greater depths m in general ceases to depend on H; some other dependence of m on H does not change the essence of the problem but affects only numerical results. Finally, the quantity l in Eq. (1.3) is the bed coefficient of the Winkler base $l = \rho g$, where g is the acceleration of gravity.

At the origin of the coordinate system the solutions of Eq. (1.3) for positive and negative x values must coincide, which implies absence of a discontinuity in the bending moment and equality of the discontinuity in transverse force to the specified force Q. Thus

$$(\dot{y}'')^{\mu}|_{+0} - (\dot{y}'')^{\mu}|_{-0} = 0; \tag{1.6}$$

$$K\{[(\dot{y}'')^{\mu}]'|_{+0} + [(\dot{y}'')^{\mu}]'|_{-0}\} = Q. \tag{1.7}$$

We note that the condition $y'|_{x=0} = 0$ in the linear case ($\mu = 1$) is automatically satisfied, while in the nonlinear case ($\mu \neq 1$) its satisfaction is incompatible with satisfaction of the more important condition (1.6).

We will construct a solution to the problem of Eqs. (1.3)-(1.7) for relatively small times t in the form of an asymptotic expansion in powers of t: for relatively large times the solution for y consists of a linear aggregate in time and an asymptotic series in powers of some exponential which vanishes as $t \to \infty$. Both solutions are applicable for moderate time values, so that together they encompass the entire time range $t \in (0, \infty)$.

2. <u>Short-Term Solution.</u> We will seek a solution of Eq. (1.3) in the form of a series

$$y = \sum_{k=1}^{\infty} y_k t^k. \tag{2.1}$$

We will first limit ourselves to two terms of the expansion. Substituting Eq. (2.1) in Eq. (1.3) we arrive at a system of two nonlinear ordinary differential equations in y_1 and y_2:

$$\mu y_1^{IV} (y_1'')^{\mu-1} + \mu(\mu-1)(y_1''')^2 (y_1'')^{\mu-2} + my_1/K = 0; \tag{2.2}$$

$$2\mu[(\mu-1) y_1^{IV} (y_1'')^{\mu-2} y_2'' + (\mu-1)(\mu-2)(y_1''')^2 (y_1'')^{\mu-3} y_2'' +$$
$$+ 2(\mu-1) y_1''' (y_1'')^{\mu-2} y_2''' + (y_1'')^{\mu-1} y_2^{IV}] + 2my_2/K + ly_1/K = 0. \tag{2.3}$$

One of the solutions of Eq. (2.2) valid everywhere but the point $x = 0$ is

$$y_1 = A(|x| + C)^{-\alpha}, \tag{2.4}$$

where

$$\alpha = 2(1+\mu)/(1-\mu), \ \alpha > 0; \tag{2.5}$$

$$A = \left\{\frac{m(1-\mu)^2}{4K\mu(1+3\mu)} \left[\frac{2(1+\mu)(3+\mu)}{(1-\mu)^2}\right]^{-\mu}\right\}^{1/(\mu-1)}. \tag{2.6}$$

Substituting Eq. (2.4) in Eq. (2.3), we find an equation for determination of y_2

$$x_1^4 y_2^{IV} + 8x_1^3 y_2''' + 12x_1^2 y_2'' + A_1 y_2 = A_2 x_1^{-\alpha},$$
$$A_1 = -8(1+\mu)(3+\mu)(1+3\mu)/(1-\mu)^4, \quad A_1 < 0, \quad x_1 = x + C, \quad A_2 = -lAA_1/2m. \tag{2.7}$$

Below we will omit the subscript of x. From Eq. (2.7) the Euler equation [4]

$$y_2^0 = x^{-1/2} \sum_{i=1}^{4} C_i^1 x^{a_i}, \quad a_i = 5/4 \pm (1 - A_1)^{1/2} \tag{2.8}$$

is homogeneous. A specific solution of Eq. (2.7) has the form

$$y_2^* = -lAx^{-\alpha}/2m(1-\mu). \tag{2.9}$$

The principle involved in creating equations of the form of Eq. (2.3) with an increased number of terms in Eq. (2.1) is simple. Thus, for the k-th term of the expansion of series (2.1)

$$(ay_k''' + a'y_k'')' + my_k/K = -ly_{k-1}/kK + F_k(y_1'', \ldots, y_{k-1}''; y_1''', \ldots, y_{k-1}'''; y_1^{IV}, \ldots, y_{k-1}^{IV}); \tag{2.10}$$

$$a = \mu(y_1'')^{\mu-1}. \tag{2.11}$$

For example, if k = 3 then

$$F_3 = (-2/3)(\mu-2)\{[a'(y_1'')^{-1}]'(y_2'')^2 + 2[a'(y_1'')^{-1}][(y_2'')^2]'\} + 2(\mu-1)[a(y_1'')^{-1}][(y_2''')^2 + y_2'' y_2^{IV}].$$

For equations of the form of Eq. (2.10) we have homogeneous Euler equations, the solution of which is given by Eq. (2.8). Knowing the form of the right side for each approximation, we can write first the particular solution, then the general one.

We will now turn to determination of the constants appearing in the general solution of the problem for short times. We will again limit ourselves to two terms in the expansion of Eq. (2.1). The solution is then given by Eqs. (2.4), (2.8), (2.9). We write values for a_i (i = 1, ..., 4) appearing in Eq. (2.8):

$$a_1 = (\alpha + 1/2) > 0, \ a_2 = -(\alpha + 1/2), \ a_3 = iB_1,$$
$$a_4 = -iB_1, \ B_1 = (\alpha^2 + \alpha - 9/4)^{1/2} > 0.$$

Considering only the real component, we obtain

$$y_2^0 = C_1^1 x^\alpha + C_2^1 x^{-(1+\alpha)} + C_3^1 x^{-1/2} \cos(B_1 \ln x).$$

The condition at infinity permits the conclusion that $C_1^1 = 0$. The general solution for this approximation is then given by:

$$y = Ax^{-\alpha} t + [C_2^1 x^{-(1+\alpha)} + C_3^1 x^{-1/2} \cos(B_1 \ln x) - lAx^{-\alpha}/2m(1-\mu)]\, t^2,$$
$$x = |x| + C.$$

Here the constants C, C_2^1 and C_3^1 are defined by the boundary conditions

$$C = [2mA/Q(\alpha-1)]^{1/(\alpha-1)},$$
$$C_2^1 = (lA/m)[\mu(1+\mu)/(1-\mu)(1+3\mu)]\, C + [(1-\mu)^2/2(1+3\mu)] \times$$
$$\times [(1/2)\cos(B_1 \ln C) + B_1 \sin(B_1 \ln C)]\, C_3 C^{1/2+\alpha},$$
$$C_3^1 = -C^{1/2-\alpha} D_1 [\alpha(1+\alpha)/(\alpha-1)] \{[2 + \alpha(1+\alpha)/(\alpha-1)] \times$$
$$\times [(1/2)\cos(B_1 \ln C) + B_1 \sin(B_1 \ln C)] - (\alpha-1)(\alpha+2)\cos(B_1 \ln C)\}^{-1},$$
$$D_1 = -lA/2m(1-\mu).$$

Increasing the number of terms used in Eq. (2.1) and analyzing the solutions obtained in a manner similar to that used for the previous approximation, it is simple to establish the relationships between the constants for the previous and next approximations at any step of the solution process:

$$C_2^k = C_2^{k-1} D_k / D_{k-1}, \quad C_3^k = C_3^{k-1} D_k / D_{k-1}, \quad k = 2, \ldots$$

We note, for example, that at $k = 3$, $D_3 = l^2 A / 6m^2(1-\mu)$, at $k = 4$, $D_4 = -Al^3(2+\alpha)[1 + (\alpha-2)^2/16]/96m^3$.

3. Long-Term Solution. We will seek a solution of Eq. (1.3) in the form of a series

$$y = y_0 + f_0 t + \sum_{k=1}^{\infty} f_k \exp(-kRt), \quad R = \text{const} > 0. \tag{3.1}$$

Since $y|_{t=0} = 0$, then

$$y_0 = -\sum_{k=1}^{\infty} f_k. \tag{3.2}$$

We will consider the first approximation. Let

$$y = y_0 + f_0 t + f_1 \exp(-Rt). \tag{3.3}$$

Substituting Eq. (3.3) in the original equation (1.3) and introducing the notation of Eq. (2.11), we obtain

$$(af_0''')' + mf_0/K = -ly_0/K; \tag{3.4}$$

$$(af_1''' + a'f_1'')' + (Rm - l)\, f_1/KR = (lf_0 t/KR)\exp(Rt). \tag{3.5}$$

We will increase the number of terms in expansion (3.1). Let $y = y_0 + f_0 t + f_1 \exp(-Rt) + f_2 \exp(-2Rt)$. We then arrive at a system of three equations for determination of f_0, f_1, and f_2, with f_0 and f_1 being found from Eqs. (3.4), (3.5), respectively, while for f_2

$$(af_2''' + a'f_2'')' + (m/K - l/2KR)\, f_2 = [R(\mu-2)/4]\{[a'(f_0'')^{-1}]'(f_1'')^2 +$$
$$+ 2[a'(f_0'')^{-1}][(f_1'')^2]'\} + (1/2)R(\mu-1)[a(f_0'')^{-1}][(f_1''')^2 + f_1 f_1^{\mathrm{IV}}]. \tag{3.6}$$

The homogeneous equations for Eqs. (3.4)-(3.6) are the same as in Sec. 2 – Euler equations with known solutions [4]. For each approximation, knowing the form of the right side of the equation, we write the particular solution, then derive the general one.

We will now turn to determination of the constants appearing in the general solution for the case of large times. We will limit ourselves to the expansion of Eq. (3.3). Then

$$y = (y_0 + f_1) + (f_0 - Rf_1)\, t + f_1 \sum_{n=2}^{k} (-Rt)^n / n! \tag{3.7}$$

We introduce the notation:

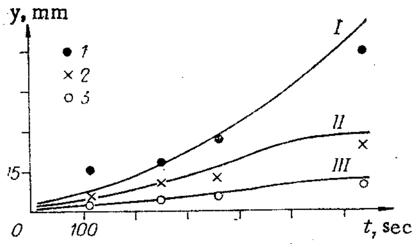

Fig. 2

$$F_0 = y_0 + f_1,\ F_1 = f_0 - Rf_1,\ F_2 = R^2 f_1/2, \ldots,\ F_k = (-1)^k R^k f_1/k!$$

We express f_0, f_1, and R in terms of F_1, F_2, F_3:

$$f_0 = F_1 - (2/3)(F_2^2/F_3), \quad R = -3F_3/F_2, \quad f_1 = (2/9)(F_2^3/F_3^2).$$

In general, for $k \geq 4$, $F_k = (3/k)(F_3/F_2)F_{k-1}$.

On the basis of Eq. (3.2) we conclude that $F_0 = 0$. Comparison of Eqs. (3.7) and (2.1) permits the conclusion that $F_1 = y_1$, $F_2 = y_2$, $F_3 = y_3$. Expressions for y_1, y_2, and y_3 were presented in Sec. 2. Thus

$$y = (F_1 + 2mF_2/l)t + [1 - \exp(-lt/m)]2m^2F_2/l^2,$$
$$F_1 = Ax^{-\alpha},\ F_2 = Dx^{-\alpha} + C_2x^{-(\alpha+1)} + C_5x^{-1/2}\cos(B_1 \ln x).$$

Thus, knowing the solution of the problem for small time values, the solution for large times can be written. The studies carried out in Secs. 2 and 3 permit the conclusion that we have determined solutions of nonlinear equation (1.3) valid for any moment of time and convenient for practical applications.

4. Full-Scale Studies. The experiments were performed on the ice coating of one of the arctic seas of the USSR. The presence of a fresh water source and low air temperatures allowed creation of ice coatings of various salinities and thicknesses. Preparation for the experiments involved creation of a stable ice coating, and determination of the temperature, thickness, salinity, and texture of the ice in the vicinity of the control area. The deflection of the natural ice coating with time was determined by simultaneous stereophotogrammetric and cine photography, and underwater photography and television. To do this a 1×1 m grid of stakes was established on the icefield, such that the stake length above the ice was 0.3 m, with 1 m showing beneath the ice. Underwater photography and television monitoring of the lower boundary of the ice coating and its texture were carried out. The image was fed to a monitor and a video tape recorder. All the photographic methods were operated synchronously.

The experiments consisted of placing weights from 2.5 to 32.5 tons on the ice and turning on all the photographic equipment to record the bending of the ice over time. For all the applied loads it was possible to determine the steady state creep and measure the deflection at various points in the ice coating. Figure 2 shows characteristic points 1-3 of the experimental studies for Q = 2.5 tons, h = 0.25 m, H = 12 m, with sensors located 1, 2, and 3 m from the load, while lines I-III show results of calculations with the expressions derived in Secs. 2 and 3. Comparison of the theoretically calculated plate deflections over coordinate x with time t with those measured in the real ice coating shows good qualitative and satisfactory quantitative agreement. In the calculations the rheological constants for the ice were taken from [5] ($B = 5.6(0.1\ \text{MPa})^{-b}\ \text{sec}^{-1}$, $b = 1.72$).

For engineering applications approximate expressions for evaluating the rheological characteristics of the ice coating can be used. We will assume that in Eq. (2.1), y_1, y_2, y_3, ... are known. We will limit ourselves to two terms of the series, considering $\lim_{x\to 0} y_1$ and $\lim_{x\to 0} y_2$. We will evaluate all terms in the expressions obtained. It is obvious that the terms containing the constant C_5 are small in comparison to the remaining ones, so we will neglect them. Finally we obtain

$$b = -2(ly_1 + 3my_2)/(ly_1 + 2my_2) \tag{4.1}$$

(where $y_2 < 0$ always). In the case considered $y_2 = -y_1 l(3\alpha - 2)/8m(\alpha - 1)$. After simple transformations it can easily be shown that the numerator in Eq. (4.1) is always negative, while the denominator is always positive.

For B we find the expression

$$B = h\left(\frac{m}{Q}\right)^2 \frac{2\alpha(\alpha+1)}{(\alpha-1)^2} y_1^3 \left[-\frac{16g}{l}\frac{\alpha^2-4}{9\alpha^2-4}\left(\frac{mh}{Q}\right)^2 y_2\right]^b.$$

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DYNAMIC DEFORMATION OF A WEDGE MADE OF AN INHOMOGENEOUS HARDENING MATERIAL

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UDC 539.374

Consideration is given to compression and bending of a plane infinite wedge at whose tip a concentrated force P(t) is applied varying with time by a special rule. The material is assumed to be incompressible, plastically inhomogeneous, and it obeys an exponential hardening rule. In essence, this material may also relate to a nonlinearly elastic, nonlinearly ductile body whose compressibility may ignored. A study is also made of the effect of external forces with which points of the body complete vibratory and monotonic movements in time. Concentrated forces are determined corresponding to the deformed state of the wedge being considered. Questions of unloading are not discussed, and therefore for the case of plastic bodies a study is also made of the stages of movement which lead to loading.

The stressed state in plastically inhomogeneous bodies under dynamic effects has been studied in [1-3, and others]. A similar analysis of dynamic problems for plastically inhomogeneous bodies is given in [4, 5]. A study of dynamic deformation questions for a plastically inhomogeneous incompressible body is of interest, particularly from the point of view of studying the effect of inertial forces on the stressed–strained state of the body.

Dynamic problems for incompressible ductile materials with axisymmetric and planar deformation have been considered in [6, 7].

1. Equations for deformation theory of plasticity for an incompressible, inhomogeneous material with an exponential hardening rule in the case of plane strain have in normal notation the following form:

differential equations of motion

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2},$$
$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r}\tau_{r\theta} = \rho \frac{\partial^2 v}{\partial t^2}; \tag{1.1}$$

relationships between stress and strain intensities

$$\varepsilon_0 = \left(\frac{\sigma_0}{K}\right)^3, \quad K = K(r, \theta),$$
$$\sigma_0 = \frac{1}{2}\sqrt{(\sigma_r - \sigma_\theta)^2 + 4\tau_{r\theta}^2}, \quad \varepsilon_0 = \sqrt{(\varepsilon_r - \varepsilon_\theta)^2 + 4\gamma_{r\theta}^2} \tag{1.2}$$

[K(r, θ) is a known function characterizing a plastically inhomogeneous material];

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